

HOLOMORPHIC SUBMANIFOLDS OF SOME HYPERCOMPLEX MANIFOLDS WITH HERMITIAN AND NORDEN METRICS

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ABSTRACT. In this paper we initiate the study of submanifolds of almost hypercomplex manifolds with Hermitian and Norden metrics. Object of investigations are holomorphic submanifolds of the hypercomplex manifolds which are locally conformally equivalent to the hyper-Kähler manifolds of the considered type. Necessary and sufficient conditions the investigated holomorphic submanifolds to be totally umbilical or totally geodesic are obtained. Examples of the examined submanifolds are constructed.

INTRODUCTION

The almost hypercomplex manifolds with Hermitian and Norden metrics have been introduced by Gribachev, Manev and Dimiev in [1]. These manifolds are equipped with an almost hypercomplex structure $H = (J_1, J_2, J_3)$ and a metric structure $G = (g, g_1, g_2, g_3)$. Here g is a neutral metric, which is Hermitian with respect to the almost complex structure J_1 of H and g is a Norden metric (known also as an anti-Hermitian metric) regarding the almost complex structures J_2 and J_3 of H . Moreover, G contains the Kähler 2-form g_1 with respect to J_1 and two associated Norden metrics g_2, g_3 with respect to J_2 and J_3 , respectively. The geometry of almost hypercomplex manifolds with Hermitian and Norden metrics has been investigated in [1, 2, 3, 4]. This type of manifolds are the only possible case to involve Norden metrics on almost hypercomplex manifolds.

In this paper we initiate the study of submanifolds of the considered manifolds. As first step in this direction we study their holomorphic submanifolds. A holomorphic submanifold M of an almost hypercomplex manifold with Hermitian and Norden metrics (\overline{M}, H, G) is a non-degenerate submanifold such that the tangent bundle is preserved by the structure H , which implies that M is also an almost hypercomplex manifold with Hermitian and Norden metrics with respect to the restrictions of the structures H and G on M .

An almost hypercomplex structure H on \overline{M} is called hypercomplex if the three almost complex structures J_1, J_2, J_3 are integrable. Objects of special interest in this work are ambient manifolds (\overline{M}, H, G) belonging to the class denoted by \mathcal{W} of hypercomplex manifolds with Hermitian and Norden metrics. This is the class of the locally conformally equivalent manifolds to the manifolds in the class \mathcal{K} consisting of the hyper-Kähler manifolds of the considered type. The class \mathcal{K} is an important subclass of \mathcal{W} , where the considered manifolds have parallel J_1, J_2, J_3 with respect to the Levi-Civita connection ∇ of g .

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The present paper is organized as follows. In Section 1 we present some definitions and facts about almost hypercomplex manifolds with Hermitian and Norden metrics. Section 2 is devoted to the study of holomorphic submanifolds of the considered manifolds belonging to the classes \mathcal{W} and \mathcal{K} . We prove that a holomorphic submanifold M of a \mathcal{W} -manifold \overline{M} is either a \mathcal{W} -manifold, or a \mathcal{K} -manifold. Moreover, we obtain necessary and sufficient conditions M to be in \mathcal{W} or \mathcal{K} in terms of the Lee 1-forms $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) of \overline{M} . We also show that a holomorphic submanifold M of a \mathcal{W} -manifold \overline{M} is either totally umbilical, or totally geodesic and find necessary and sufficient conditions for this, expressed by conditions for $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$). We obtain that every holomorphic submanifold of a \mathcal{K} -manifold is a totally geodesic \mathcal{K} -manifold and every holomorphic \mathcal{K} -submanifold of a \mathcal{W} -manifold is totally umbilical. In Section 3 we construct examples of the studied submanifolds.

1. PRELIMINARIES

A $4n$ -dimensional differentiable manifold (M, H) is called an *almost hypercomplex manifold* [5] if it is equipped with an *almost hypercomplex structure* $H = (J_1, J_2, J_3)$, which is a triple of almost complex structures having the properties:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I$$

for all cyclic permutations (α, β, γ) of $(1, 2, 3)$ and the identity I .

Let g be a pseudo-Riemannian metric on (M, H) which is Hermitian with respect to J_1 and g is a Norden metric with respect to J_2 and J_3 , i.e.

$$(1.1) \quad g(J_1 X, J_1 Y) = -g(J_2 X, J_2 Y) = -g(J_3 X, J_3 Y) = g(X, Y), \quad X, Y \in TM.$$

The associated bilinear forms g_1, g_2 and g_3 are determined by

$$(1.2) \quad g_1(X, Y) = g(J_1 X, Y), \quad g_2(X, Y) = g(J_2 X, Y), \quad g_3(X, Y) = g(J_3 X, Y).$$

According to (1.1) and (1.2) the metric g and the associated bilinear forms g_2 and g_3 are necessarily pseudo-Riemannian metrics of neutral signature $(2n, 2n)$ and g_1 is the known Kähler 2-form Φ with respect to J_1 .

Differentiable $4n$ -dimensional manifolds M equipped with structures $(H, G) = (J_1, J_2, J_3, g, g_1, g_2, g_3)$ are studied in [1, 4] and [2, 3] (under the name *almost hypercomplex pseudo-Hermitian manifolds* and *almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics*, respectively). In this paper we refer to (M, H, G) as an *almost hypercomplex manifold with Hermitian and Norden metrics*.

The $(0, 3)$ -tensors $F_\alpha(X, Y, Z) = g((\nabla_X J_\alpha)Y, Z)$, ($\alpha = 1, 2, 3$), where ∇ is the Levi-Civita connection generated by g , are called *fundamental tensors* of (M, H, G) . It is well known that the almost hypercomplex structure $H = (J_1, J_2, J_3)$ is a hypercomplex structure if the Nijenhuis tensor $N_\alpha(X, Y)$ for J_α vanishes for each $\alpha = 1, 2, 3$. Moreover, an almost hypercomplex structure H is hypercomplex if and only if two of the tensors N_α vanish [5].

Let us remark that (M, H, G) is an indefinite almost Hermitian manifold with respect to J_1 and it is an almost complex manifold with Norden metric with respect to J_2 and J_3 . The basic classifications of the almost complex manifolds with Hermitian metric and with Norden metric are given in [6] and [7], respectively.

Let us assume that (M, H, G) belongs to the class \mathcal{W}_4 from the Gray-Hervella classification, which is a subclass of the class of Hermitian manifolds and it is the

class of locally conformal equivalent manifolds to the Kähler manifolds. Then the almost complex structure J_1 is integrable and F_1 is given by

$$(1.3) \quad F_1(X, Y, Z) = \frac{1}{2(2n-1)} [g(X, Y)\theta_1(Z) - g(X, Z)\theta_1(Y) - g(X, J_1Y)\theta_1(J_1Z) + g(X, J_1Z)\theta_1(J_1Y)],$$

where the Lee form θ_1 is defined by $\theta_1(Z) = g^{ij}F_1(e_i, e_j, Z)$ for a basis $\{e_i\}$, $(i = 1, \dots, 4n)$ and (g^{ij}) is the inverse matrix of the matrix (g_{ij}) of the metric g .

One of the basic classes of the integrable almost complex manifolds with Norden metric is \mathcal{W}_1 . It is a subclass of the integrable (almost) complex manifolds with Norden metric and it is the class of the locally conformal equivalent manifolds to the Kähler manifolds with Norden metric. If (M, H, G) belongs to $\mathcal{W}_1(J_\alpha)$, then J_α ($\alpha = 2, 3$) is integrable and the following equality holds

$$(1.4) \quad F_\alpha(X, Y, Z) = \frac{1}{4n} [g(X, Y)\theta_\alpha(Z) + g(X, Z)\theta_\alpha(Y) + g(X, J_\alpha Y)\theta_\alpha(J_\alpha Z) + g(X, J_\alpha Z)\theta_\alpha(J_\alpha Y)],$$

where the Lee form θ_α is defined by $\theta_\alpha(Z) = g^{ij}F_\alpha(e_i, e_j, Z)$ ($\alpha = 2, 3$) for a basis $\{e_i\}$, $(i = 1, \dots, 4n)$.

The class \mathcal{W} , studied in [1], consists of all hypercomplex manifolds with Hermitian and Norden metrics such that F_1 satisfies (1.3) and F_2, F_3 satisfy (1.4). A manifold (M, H, G) belonging to the class \mathcal{W} is called briefly a \mathcal{W} -manifold.

It is known [1] that necessary and sufficient conditions (M, H, G) to be a \mathcal{W} -manifold are

$$(1.5) \quad \theta_\alpha \circ J_\alpha = -\frac{2n}{2n-1}\theta_1 \circ J_1, \quad \alpha = 2, 3.$$

According to [1], an almost hypercomplex manifold with Hermitian and Norden metrics is called a hyper-Kähler manifold of the considered type if $\nabla J_\alpha = 0$ ($\alpha = 1, 2, 3$) with respect to the Levi-Civita connection generated by g . The class of these manifolds is denoted by \mathcal{K} in [1] and thus we call them \mathcal{K} -manifolds. It is clear that for the \mathcal{K} -manifolds the conditions $F_\alpha = 0$ ($\alpha = 1, 2, 3$) hold and therefore $\theta_\alpha = 0$ ($\alpha = 1, 2, 3$) and \mathcal{K} is a subclass of \mathcal{W} .

2. HOLOMORPHIC SUBMANIFOLDS OF \mathcal{W} -MANIFOLDS

A $4m$ -dimensional submanifold M of a $4n$ -dimensional ($m < n$) almost hypercomplex manifold with Hermitian and Norden metrics (\overline{M}, H, G) is said to be a *holomorphic submanifold* if the tangent bundle TM is preserved by J_α ($\alpha = 1, 2, 3$), i.e. $J_\alpha(T_p M) = T_p M$ for all $p \in M$ and the restriction of the metric g on TM has a maximal rank.

We denote the restrictions of g and J_α ($\alpha = 1, 2, 3$) on TM by the same letters. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of (\overline{M}, H, G) and its submanifold M , respectively. Then the Gauss-Weingarten formulae are given by

$$(2.1) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \overline{\nabla}_X N = -A_N X + D_X N,$$

where $X, Y \in TM$, $N \in TM^\perp$, h is the second fundamental form of M , A_N is the shape operator in the direction of a normal vector field N and D is the normal connection. The shape operator and the second fundamental form are related as usually by $g(A_N X, Y) = g(h(X, Y), N)$.

A submanifold M is said to be *totally umbilical* if $h(X, Y) = g(X, Y)C$, where $C = \frac{1}{4m} \text{trace } h$ is the *mean curvature vector* of M . If $h(X, Y) = 0$, then M is said to be *totally geodesic*.

Let (\overline{M}, H, G) be a \mathcal{W} -manifold. We denote by $\overline{F}_\alpha, \overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) the fundamental tensors and the Lee forms of \overline{M} . Then the equalities (1.3), (1.4) and (1.5) hold. If p_α are the covectors of $\overline{\theta}_\alpha$, i.e. $\overline{\theta}_\alpha(Z) = g(Z, p_\alpha)$ for any $Z \in T\overline{M}$ and $\alpha = 1, 2, 3$, from (1.5) we obtain

$$(2.3) \quad J_\alpha p_\alpha = \frac{2n}{2n-1} J_1 p_1, \quad \alpha = 2, 3.$$

Let M be a submanifold of (\overline{M}, H, G) . Then for every p_α the following decomposition is valid

$$(2.4) \quad p_\alpha = p_\alpha^\top + p_\alpha^\perp,$$

where $p_\alpha^\top \in TM$ and $p_\alpha^\perp \in TM^\perp$. Taking into account (2.4) we conclude that $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanishes on TM (resp. TM^\perp) if and only if p_α^\top (resp. p_α^\perp) vanishes.

Let M be a holomorphic submanifold of a \mathcal{W} -manifold (\overline{M}, H, G) . Then from (1.5) it follows that $\overline{\theta}_1, \overline{\theta}_2$ and $\overline{\theta}_3$ are either all non-zero on TM (resp. TM^\perp), or all vanish on TM (resp. TM^\perp). Moreover, having in mind (2.4) and the fact that $\overline{\theta}_\alpha \neq 0$ ($\alpha = 1, 2, 3$) for a \mathcal{W} -manifold which is not a \mathcal{K} -manifold, we establish that if all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish on TM (resp. TM^\perp), then all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero on TM^\perp (resp. TM).

Theorem 2.1. *Let M be a holomorphic submanifold of a \mathcal{W} -manifold (\overline{M}, H, G) . Then we get*

$$(2.5) \quad h(X, Y) = \frac{1}{2(2n-1)} g(X, Y) J_1(p_1^\perp) = \frac{1}{4n} g(X, Y) J_\alpha(p_\alpha^\perp), \quad \alpha = 2, 3,$$

$$(2.6) \quad (\nabla_X J_1)Y = \frac{1}{2(2n-1)} [g(X, Y)p_1^\top - \overline{\theta}_1(Y)X + g(X, J_1 Y)J_1(p_1^\top) - \overline{\theta}_1(J_1 Y)J_1 X],$$

$$(2.7) \quad (\nabla_X J_\alpha)Y = \frac{1}{4n} [g(X, Y)p_\alpha^\top + \overline{\theta}_\alpha(Y)X + g(X, J_\alpha Y)J_\alpha(p_\alpha^\top) + \overline{\theta}_\alpha(J_\alpha Y)J_\alpha X], \quad \alpha = 2, 3,$$

$$(2.8) \quad A_{J_1 N}X = J_1(A_N X) + \frac{1}{2(2n-1)} [\overline{\theta}_1(N)X + \overline{\theta}_1(J_1 N)J_1 X],$$

$$(2.9) \quad A_{J_\alpha N}X = J_\alpha(A_N X) - \frac{1}{4n} [\overline{\theta}_\alpha(N)X + \overline{\theta}_\alpha(J_\alpha N)J_\alpha X], \quad \alpha = 2, 3,$$

$$(2.10) \quad D_X J_\alpha N = J_\alpha(D_X N), \quad \alpha = 1, 2, 3,$$

where $X, Y \in TM$ and $N \in TM^\perp$.

Proof. Using (2.1), we obtain

$$(2.11) \quad (\overline{\nabla}_X J_\alpha)Y = (\nabla_X J_\alpha)Y + h(X, J_\alpha Y) - J_\alpha h(X, Y), \quad \alpha = 1, 2, 3.$$

From (1.3) and (1.4) for \overline{F}_1 and \overline{F}_α ($\alpha = 2, 3$), respectively, we get

$$(2.12) \quad (\overline{\nabla}_X J_1)Y = \frac{1}{2(2n-1)} [g(X, Y)p_1 - \overline{\theta}_1(Y)X + g(X, J_1 Y)J_1 p_1 - \overline{\theta}_1(J_1 Y)J_1 X],$$

$$(2.13) \quad (\bar{\nabla}_X J_\alpha)Y = \frac{1}{4n} [g(X, Y)p_\alpha + \bar{\theta}_\alpha(Y)X + g(X, J_\alpha Y)J_\alpha p_\alpha + \bar{\theta}_\alpha(J_\alpha Y)J_\alpha X].$$

By substituting (2.12) in (2.11) and taking into account that M is holomorphic, we obtain (2.6) and

$$(2.14) \quad h(X, J_1 Y) - J_1 h(X, Y) = \frac{1}{2(2n-1)} [g(X, Y)p_1^\perp + g(X, J_1 Y)J_1(p_1^\perp)].$$

Analogously, using (2.11) and (2.13), we get (2.7) and

$$(2.15) \quad h(X, J_\alpha Y) - J_\alpha h(X, Y) = \frac{1}{4n} [g(X, Y)p_\alpha^\perp + g(X, J_\alpha Y)J_\alpha(p_\alpha^\perp)], \quad \alpha = 2, 3.$$

We replace X, Y by Y, X in (2.14) and (2.15) and we find that

$$(2.16) \quad h(Y, J_1 X) - J_1 h(X, Y) = \frac{1}{2(2n-1)} [g(X, Y)p_1^\perp - g(X, J_1 Y)J_1(p_1^\perp)],$$

$$(2.17) \quad h(Y, J_\alpha X) - J_\alpha h(X, Y) = \frac{1}{4n} [g(X, Y)p_\alpha^\perp + g(X, J_\alpha Y)J_\alpha(p_\alpha^\perp)], \quad \alpha = 2, 3.$$

Subtracting (2.16) and (2.17) from (2.14) and (2.15), respectively, we obtain

$$h(X, J_1 Y) - h(J_1 X, Y) = \frac{1}{2n-1} g(X, J_1 Y)J_1(p_1^\perp),$$

$$h(X, J_\alpha Y) = h(J_\alpha X, Y), \quad \alpha = 2, 3.$$

From the latter two equalities it follows

$$(2.18) \quad h(J_1 X, J_1 Y) + h(X, Y) = \frac{1}{2n-1} g(X, Y)J_1(p_1^\perp),$$

$$(2.19) \quad h(J_\alpha X, J_\alpha Y) = -h(X, Y), \quad \alpha = 2, 3.$$

We substitute $J_2 X$ and $J_3 Y$ for X and Y in (2.18), respectively and taking into account the equality (2.19) we find the first equality in (2.5). The rest equalities in (2.5) hold because of (2.3). By using (2.2) we obtain

$$(2.20) \quad (\bar{\nabla}_X J_\alpha)N = -A_{J_\alpha N}X + J_\alpha(A_N X) + D_X J_\alpha N - J_\alpha(D_X N), \quad \alpha = 1, 2, 3.$$

From the conditions (1.3) and (1.4) for \bar{F}_1 and \bar{F}_α ($\alpha = 2, 3$), respectively, it follows

$$(2.21) \quad (\bar{\nabla}_X J_1)N = -\frac{1}{2(2n-1)} [\bar{\theta}_1(N)X + \bar{\theta}_1(J_1 N)J_1 X],$$

$$(2.22) \quad (\bar{\nabla}_X J_\alpha)N = -\frac{1}{4n} [\bar{\theta}_\alpha(N)X + \bar{\theta}_\alpha(J_\alpha N)J_\alpha X], \quad \alpha = 2, 3.$$

Finally, (2.20), (2.21) and (2.22) imply (2.8), (2.9), (2.10). \square

Theorem 2.2. *Let M be a holomorphic submanifold of a \mathcal{W} -manifold (\bar{M}, H, G) . Then M is a \mathcal{W} -manifold (resp. a \mathcal{K} -manifold) if and only if all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero (resp. all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish) on TM .*

Proof. It is clear that a holomorphic submanifold of an almost hypercomplex manifold with Hermitian and Norden metrics (\bar{M}, H, G) supplied with the restrictions of the structures $(H, G) = (J_1, J_2, J_3, g, g_1, g_2, g_3)$ (the restrictions are denoted by the same letters) is also an almost hypercomplex manifold with Hermitian and Norden

metrics. Using (2.6) and (2.7), for the fundamental tensors F_α ($\alpha = 1, 2, 3$) of M we get the following:

$$(2.23) \quad \begin{aligned} F_1(X, Y, Z) &= \frac{1}{2(2m-1)} [g(X, Y)\theta_1(Z) - g(X, Z)\theta_1(Y) \\ &\quad - g(X, J_1 Y)\theta_1(J_1 Z) + g(X, J_1 Z)\theta_1(J_1 Y)], \\ F_\alpha(X, Y, Z) &= \frac{1}{4m} [g(X, Y)\theta_\alpha(Z) + g(X, Z)\theta_\alpha(Y) \\ &\quad + g(X, J_\alpha Y)\theta_\alpha(J_\alpha Z) + g(X, J_\alpha Z)\theta_\alpha(J_\alpha Y)], \end{aligned}$$

where $X, Y, Z \in TM$ and $\theta_\alpha(Z) = g^{ij}F_\alpha(e_i, e_j, Z)$ ($\alpha = 1, 2, 3$) for an arbitrary basis $\{e_i\}$, ($i = 1, \dots, 4m$) of TM . Moreover, the Lee forms θ_α of M and $\bar{\theta}_\alpha$ of \bar{M} are related by the equalities

$$(2.24) \quad \theta_1(Z) = \frac{2m-1}{2n-1}\bar{\theta}_1(Z), \quad \theta_\alpha(Z) = \frac{m}{n}\bar{\theta}_\alpha(Z), \quad \alpha = 2, 3.$$

From the latter results it follows that θ_1 , θ_2 and θ_3 are either all non-zero, or all vanish. In the first case, having in mind (1.3), (1.4) and (2.23), we obtain that M is a \mathcal{W} -manifold. In the case when all θ_α ($\alpha = 1, 2, 3$) vanish, from (2.23) we get that $F_\alpha = 0$ ($\alpha = 1, 2, 3$). Therefore, M is a \mathcal{K} -manifold.

Conversely, let M be a \mathcal{W} -manifold (resp. a \mathcal{K} -manifold). Then all θ_α ($\alpha = 1, 2, 3$) are non-zero (resp. all θ_α ($\alpha = 1, 2, 3$) vanish) and the equalities (2.24) imply all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero (resp. all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish) on TM . \square

As an immediate consequence from Theorem 2.2 and (2.5) we state

Corollary 2.1. *Every holomorphic submanifold M of a \mathcal{K} -manifold (\bar{M}, H, G) is a totally geodesic \mathcal{K} -manifold.*

Theorem 2.3. *Let M be a holomorphic submanifold of a \mathcal{W} -manifold (\bar{M}, H, G) . Then M is totally umbilical (resp. totally geodesic) if and only if all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero (resp. all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish) on TM^\perp .*

Proof. Using (2.5) we get

$$\text{trace } h = \frac{2m}{2n-1}J_1(p_1^\perp) = \frac{m}{n}J_\alpha(p_\alpha^\perp), \quad \alpha = 2, 3.$$

Hence for the mean curvature vector C we have

$$(2.25) \quad C = \frac{1}{2(2n-1)}J_1(p_1^\perp) = \frac{1}{4n}J_\alpha(p_\alpha^\perp), \quad \alpha = 2, 3.$$

According to (2.5) and (2.25) we obtain

$$(2.26) \quad h(X, Y) = g(X, Y)C, \quad X, Y \in TM.$$

Now, from (2.26) it follows that M is either totally umbilical (when $C \neq 0$), or totally geodesic (when $C = 0$). Let M be totally umbilical (resp. totally geodesic). Then the equalities (2.25) imply that all p_α^\perp ($\alpha = 1, 2, 3$) are non-zero (resp. all p_α^\perp ($\alpha = 1, 2, 3$) vanish). Hence we obtain that all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero (resp. all $\bar{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish) on TM^\perp . We prove the sufficient condition of the theorem using (2.25). \square

Using Theorem 2.2 and Theorem 2.3 we obtain the following corollaries.

Corollary 2.2. *Every totally geodesic holomorphic submanifold M of a \mathcal{W} -manifold (\bar{M}, H, G) is a \mathcal{W} -manifold.*

Corollary 2.3. *Every holomorphic \mathcal{K} -submanifold M of a \mathcal{W} -manifold (\overline{M}, H, G) is totally umbilical.*

Corollary 2.4. *Let M be a holomorphic \mathcal{W} -submanifold of a \mathcal{W} -manifold (\overline{M}, H, G) . If all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero (resp. all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish) on TM^\perp , then M is totally umbilical (resp. totally geodesic).*

3. EXAMPLES OF HOLOMORPHIC SUBMANIFOLDS

Example 3.1. Let we consider the vector space

$$\mathbb{R}^{4n} = \{p = (x^1, \dots, x^n, y^1, \dots, y^n, u^1, \dots, u^n, v^1, \dots, v^n) \mid x^i, y^i, u^i, v^i \in \mathbb{R}\}.$$

In [1] a hypercomplex structure $H = (J_1, J_2, J_3)$ and a pseudo-Riemannian metric g of signature $(2n, 2n)$ are defined on \mathbb{R}^{4n} as follows:

$$(3.1) \quad \begin{aligned} J_1 \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial y^i}, & J_1 \frac{\partial}{\partial y^i} &= -\frac{\partial}{\partial x^i}, & J_1 \frac{\partial}{\partial u^i} &= -\frac{\partial}{\partial v^i}, & J_1 \frac{\partial}{\partial v^i} &= \frac{\partial}{\partial u^i}; \\ J_2 \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial u^i}, & J_2 \frac{\partial}{\partial y^i} &= \frac{\partial}{\partial v^i}, & J_2 \frac{\partial}{\partial u^i} &= -\frac{\partial}{\partial x^i}, & J_2 \frac{\partial}{\partial v^i} &= -\frac{\partial}{\partial y^i}; \\ J_3 \frac{\partial}{\partial x^i} &= -\frac{\partial}{\partial v^i}, & J_3 \frac{\partial}{\partial y^i} &= \frac{\partial}{\partial u^i}, & J_3 \frac{\partial}{\partial u^i} &= -\frac{\partial}{\partial y^i}, & J_3 \frac{\partial}{\partial v^i} &= \frac{\partial}{\partial x^i}, \end{aligned}$$

$$(3.2) \quad g(X, X) = \delta_{ij} (-p^i p^j - q^i q^j + r^i r^j + s^i s^j),$$

where $X = p^i \frac{\partial}{\partial x^i} + q^i \frac{\partial}{\partial y^i} + r^i \frac{\partial}{\partial u^i} + s^i \frac{\partial}{\partial v^i}$, ($i = 1, 2, \dots, n$) is an arbitrary vector field and δ_{ij} is the Kronecker delta.

From (3.1) and (3.2) it follows that the metric g is Hermitian with respect to J_1 and g is a Norden metric with respect to J_2 and J_3 . According to (3.2) the components g_{ij} of the matrix of g with respect to the local basis $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i} \right\}$, ($i = 1, 2, \dots, n$) are constants. Therefore the Levi-Civita connection $\overline{\nabla}$ of the metric g is flat. Moreover, having in mind (3.1), it is easy to check that $\overline{\nabla} J_\alpha = 0$, ($\alpha = 1, 2, 3$). Hence, (\mathbb{R}^{4n}, H, G) is a \mathcal{K} -manifold.

Let M be a submanifold of (\mathbb{R}^{4n}, H, G) given by the following immersion:

$$\begin{aligned} &\varphi(x^1, \dots, x^m, y^1, \dots, y^m, u^1, \dots, u^m, v^1, \dots, v^m) \\ &= (x^1, \dots, x^m, x^{m+1}, \dots, x^n, y^1, \dots, y^m, y^{m+1}, \dots, y^n, \\ &\quad u^1, \dots, u^m, u^{m+1}, \dots, u^n, v^1, \dots, v^m, v^{m+1}, \dots, v^n). \end{aligned}$$

Identifying the point $(x^1, \dots, x^n, y^1, \dots, y^n, u^1, \dots, u^n, v^1, \dots, v^n)$ in \mathbb{R}^{4n} with its position vector Z , we obtain that the tangent bundle TM of M is spanned by

$$\left\{ \frac{\partial Z}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad \frac{\partial Z}{\partial y^i} = \frac{\partial}{\partial y^i}, \quad \frac{\partial Z}{\partial u^i} = \frac{\partial}{\partial u^i}, \quad \frac{\partial Z}{\partial v^i} = \frac{\partial}{\partial v^i} \right\}, \quad i = 1, 2, \dots, m.$$

From (3.1) and (3.2) it follows that the complex structures J_α ($\alpha = 1, 2, 3$) preserve TM and the submanifold M is non-degenerate. Thus, M is a $4m$ -dimensional holomorphic submanifold of (\mathbb{R}^{4n}, H, G) . The normal bundle TM^\perp is spanned by

$$\left\{ \frac{\partial}{\partial x^{m+j}}, \quad \frac{\partial}{\partial y^{m+j}}, \quad \frac{\partial}{\partial u^{m+j}}, \quad \frac{\partial}{\partial v^{m+j}} \right\}, \quad j = 1, \dots, n - m.$$

Let X be a tangent vector field and N be a normal vector field such that

$$N = a^j \frac{\partial}{\partial x^{m+j}} + b^j \frac{\partial}{\partial y^{m+j}} + c^j \frac{\partial}{\partial u^{m+j}} + d^j \frac{\partial}{\partial v^{m+j}}, \quad j = 1, \dots, n - m.$$

Then, taking into account that $\overline{\nabla}$ is flat, we find for $j = 1, \dots, n - m$

$$\overline{\nabla}_X N = (Xa^j) \frac{\partial}{\partial x^{m+j}} + (Xb^j) \frac{\partial}{\partial y^{m+j}} + (Xc^j) \frac{\partial}{\partial u^{m+j}} + (Xd^j) \frac{\partial}{\partial v^{m+j}},$$

which means that $A_N X = 0$. Hence $h(X, Y) = 0$, i.e. M is totally geodesic.

Example 3.2. Let $(M, H, G) = (J_1, J_2, J_3, g, g_1, g_2, g_3)$ and $(M', H', G') = (J'_1, J'_2, J'_3, g', g'_1, g'_2, g'_3)$ be almost hypercomplex manifolds with Hermitian and Norden metrics of dimension $4m$ and $4m'$, respectively. Let $\overline{M} = M \times M'$ be the product manifold of M and M' . Hence, \overline{M} is a $4(m+m')$ -dimensional differentiable manifold and for any vector field \overline{X} on \overline{M} the following decomposition is valid

$$\overline{X} = (X, X') = X + X',$$

where $X \in TM$ and $X' \in TM'$. Following [8], we define three almost complex structures $\overline{J}_1, \overline{J}_2, \overline{J}_3$ and a metric \overline{g} on \overline{M} by

$$(3.3) \quad \overline{J}_\alpha(X, X') = (J_\alpha X, J'_\alpha X'), \quad \alpha = 1, 2, 3,$$

$$(3.4) \quad \overline{g}((X, X'), (Y, Y')) = g(X, Y) + g'(X', Y')$$

for arbitrary $(X, X'), (Y, Y') \in T\overline{M}$. According to (3.4) any vector fields $X \in TM$ and $X' \in TM'$ are mutually orthogonal.

It is easy to check that $\overline{H} = (\overline{J}_1, \overline{J}_2, \overline{J}_3)$ is an almost hypercomplex structure on \overline{M} and \overline{g} satisfies (1.1). Thus $(\overline{M}, \overline{H}, \overline{G})$ is an almost hypercomplex manifold with Hermitian and Norden metrics, where by \overline{G} is denoted the structure $(\overline{g}, \overline{g}_1, \overline{g}_2, \overline{g}_3)$. Let $\overline{\nabla}$, ∇ and ∇' be the Levi-Civita connections of the metrics \overline{g} , g and g' , respectively. By \overline{F}_α , F_α and F'_α ($\alpha = 1, 2, 3$) we denote the fundamental tensors on \overline{M} , M and M' , respectively, i.e.

$$\begin{aligned} \overline{F}_\alpha(\overline{X}, \overline{Y}, \overline{Z}) &= \overline{g}((\overline{\nabla}_{\overline{X}} \overline{J}_\alpha) \overline{Y}, \overline{Z}); \quad \alpha = 1, 2, 3; \quad \overline{X}, \overline{Y}, \overline{Z} \in T\overline{M}, \\ F_\alpha(X, Y, Z) &= g((\nabla_X J_\alpha) Y, Z); \quad \alpha = 1, 2, 3; \quad X, Y, Z \in TM, \\ F'_\alpha(X', Y', Z') &= g'((\nabla_{X'} J'_\alpha) Y', Z'); \quad \alpha = 1, 2, 3; \quad X', Y', Z' \in TM'. \end{aligned}$$

In [8, 9] it is shown that for \overline{F}_α , F_α and F'_α ($\alpha = 1, 2, 3$) and for the corresponding Lee forms $\overline{\theta}_\alpha$, θ_α and θ'_α ($\alpha = 1, 2, 3$) the following interrelations hold

$$(3.5) \quad \overline{F}_\alpha(\overline{X}, \overline{Y}, \overline{Z}) = F_\alpha(X, Y, Z) + F'_\alpha(X', Y', Z'), \quad \alpha = 1, 2, 3,$$

$$(3.6) \quad \overline{\theta}_\alpha(\overline{Z}) = \theta_\alpha(Z) + \theta'_\alpha(Z'), \quad \alpha = 1, 2, 3,$$

where $\overline{X}, \overline{Y}, \overline{Z} \in T\overline{M}$, $X, Y, Z \in TM$, $X', Y', Z' \in TM'$.

Now, we consider the case when M is a \mathcal{K} -manifold and M' is a \mathcal{W} -manifold. Then by using (3.5) we obtain $\overline{F}_\alpha(\overline{X}, \overline{Y}, \overline{Z}) = F'_\alpha(X', Y', Z')$, which implies that $\overline{F}_\alpha(X', Y', Z') = F'_\alpha(X', Y', Z')$, ($\alpha = 1, 2, 3$). Thus we get

$$(3.7) \quad \overline{F}_\alpha(\overline{X}, \overline{Y}, \overline{Z}) = \overline{F}_\alpha(X', Y', Z') = F'_\alpha(X', Y', Z'), \quad \alpha = 1, 2, 3.$$

Taking into account that M is a \mathcal{W} -manifold, (3.6) and (3.7) we obtain that \overline{M} is a \mathcal{W} -manifold.

Let $(U \times V, x^1, \dots, x^{4m}, y^1, \dots, y^{4m'})$, (U, x^1, \dots, x^{4m}) and $(V, y^1, \dots, y^{4m'})$ be coordinate neighborhoods on \overline{M} , M and M' , respectively. Then M and M' are submanifolds of \overline{M} , given by the natural imbedding maps

$$\begin{aligned} i_1(x^1, \dots, x^{4m}) &= (x^1, \dots, x^{4m}, y^1, \dots, y^{4m'}), \\ i_2(y^1, \dots, y^{4m'}) &= (x^1, \dots, x^{4m}, y^1, \dots, y^{4m'}), \end{aligned}$$

respectively. From (3.3) it follows that the submanifolds M and M' are holomorphic. Moreover, TM^\perp (resp. TM'^\perp) coincides with TM' (resp. TM).

Since M is a holomorphic \mathcal{K} -submanifold of a \mathcal{W} -manifold \overline{M} , from Corollary 2.3 it follows that M is totally umbilical. Indeed, (3.6) imply that all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) are non-zero on TM^\perp . Then, according to Theorem 2.3, M is totally umbilical.

The holomorphic submanifold M' of a \mathcal{W} -manifold \overline{M} is a \mathcal{W} -manifold. Then from (3.6) we have $\overline{\theta}_\alpha(Z) = \theta_\alpha(Z) = 0$ ($\alpha = 1, 2, 3$). The last equalities mean that all $\overline{\theta}_\alpha$ ($\alpha = 1, 2, 3$) vanish on TM'^\perp . Finally, from Corollary 2.4 it follows that M' is totally geodesic.

Remark 3.1. A series of explicit examples of hypercomplex manifolds with Hermitian and Norden metrics belonging to the classes \mathcal{K} and \mathcal{W} are given in [4] and [2], respectively.

REFERENCES

- [1] Gribachev K., M. Manev, S. Dimiev (2003) On the almost hypercomplex pseudo-Hermitian manifolds. In: Trends in Compl. Anal., Diff. Geom. and Math. Phys. (Ed. S. Dimiev and K. Sekigawa), World Sci. Publ., 51–62.
- [2] Manev M. (2011) A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics, J. Geom. Phys., **61**, 248–259.
- [3] Manev M., K. Gribachev (2011) A connection with parallel totally skew-symmetric torsion on a class of almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics, Int. J. Geom. Methods Mod. Phys., **8**, No. 1, 115–131.
- [4] Manev M., K. Sekigawa (2005) Some four-dimensional almost hypercomplex pseudo-Hermitian manifolds. In: Contemporary Aspects in Compl. Anal., Diff. Geom. and Math. Phys. (Ed. S. Dimiev and K. Sekigawa), World Sci. Publ., 174–186.
- [5] Alekseevsky D. V., S. Marchiafava (1996) Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Appl., **CLXXI**, No. IV, 205–273.
- [6] Gray A., L. M. Hervella (1980) The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., **CXXIII**, No. IV, 35–58.
- [7] Ganchev G., A. Borisov (1986) Note on the almost complex manifolds with a Norden metric, C. R. Acad. Bulgare Sci., **39**, No. 5, 31–34.
- [8] Kanemaki S. (1984) On quasi-Sasakian manifolds, Differential geometry, Banach Center Publ., Warsaw, **12**.
- [9] Nakova G. (1999) Four-dimensional submanifolds of seven-dimensional almost contact manifolds with B-metric, Aspects of Compl. Anal., Diff. Geom., Math. Phys. and Appl., World Sci. Publ., 148–159.

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